Copyright Notice

Classification

Motivations

Classification

Question	Answer "y"	
Is this email spam?	no	yes
Is the transaction fraudulent?	no	yes
Is the tumor malignant?	no	yes

y can only be one of two values

"binary classification"

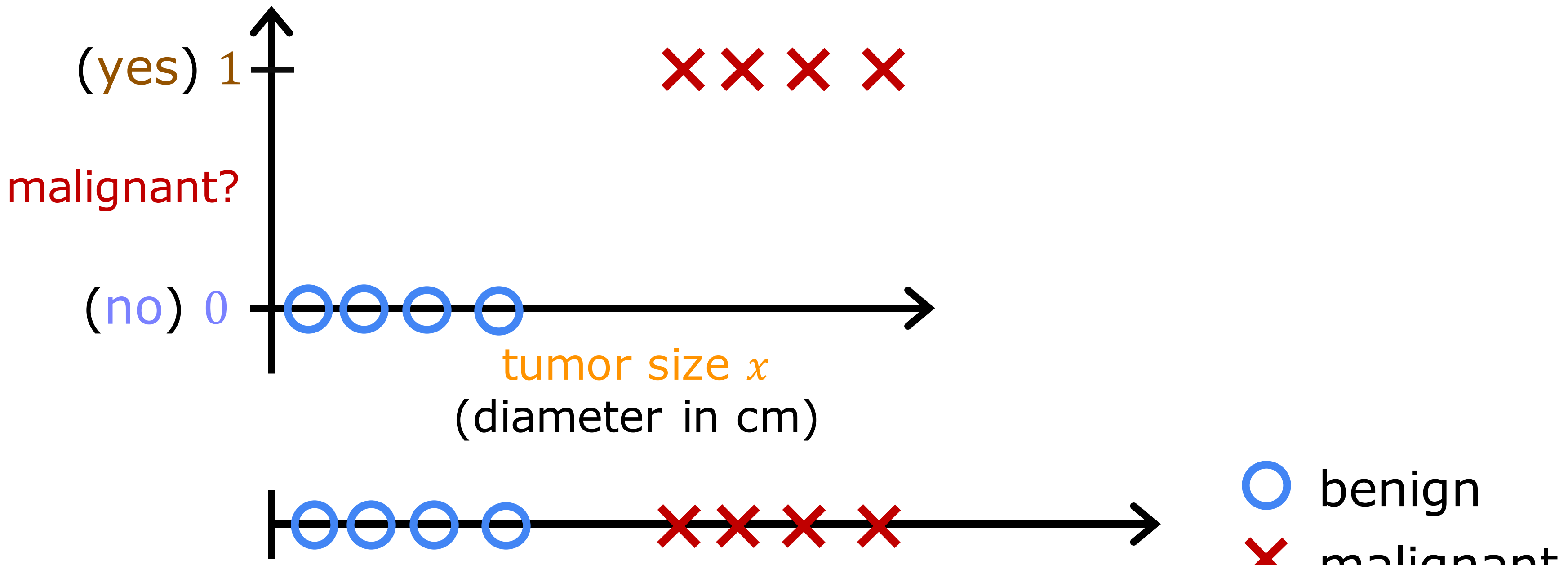
class = category

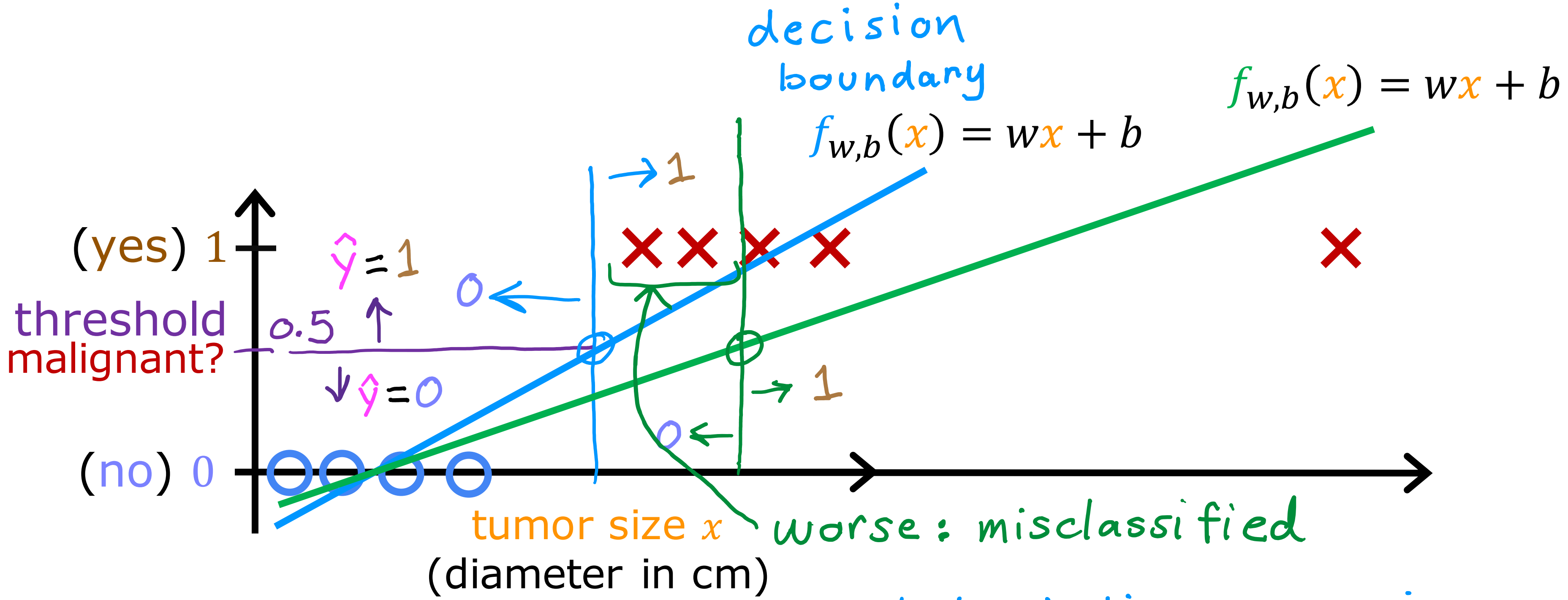
false	true
0	1

useful for classification

"negative class" ≠ "bad"
absence

"positive class" ≠ "good"
presence


(yes) 1
malignant?
(no) 0
tumor size x
(diameter in cm)
benign
malignant



if $f_{w,b}(x) < 0.5 \rightarrow \hat{y} = 0$

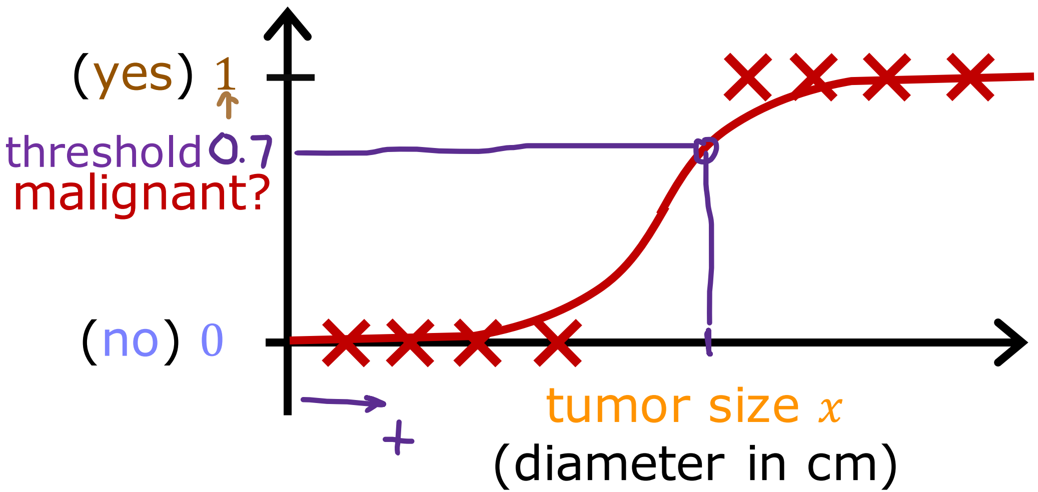
if $f_{w,b}(x) \geq 0.5 \rightarrow \hat{y} = 1$

next: logistic regression

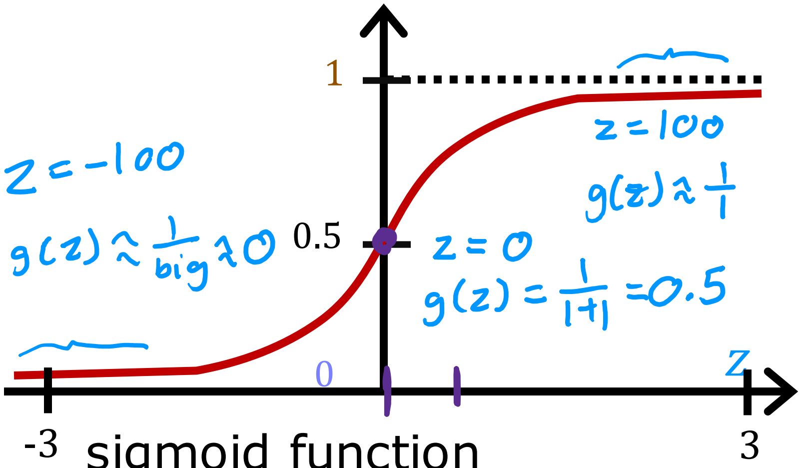
classification

Classification

Logistic Regression



Want outputs between 0 and 1



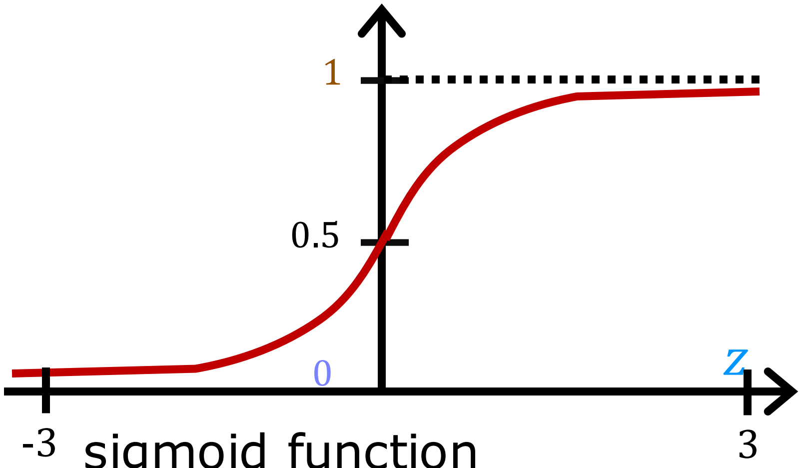
sigmoid function

logistic function

outputs between 0 and 1

$$g(z) = \frac{1}{1+e^{-z}} \qquad 0 < g(z) < 1$$

Want outputs between 0 and 1



sigmoid function

logistic function

outputs between 0 and 1

$$g(z) = \frac{1}{1+e^{-z}} \qquad 0 < g(z) < 1$$

$f_{\vec{w},b}(\vec{x})$

$\vec{w} \cdot \vec{x} + b$

↓

z

↓

$$g(z) = \frac{1}{1+e^{-z}}$$

$$f_{\vec{w},b}(\vec{x}) = g(\underbrace{\vec{w} \cdot \vec{x} + b}_{z}) = \frac{1}{1 + e^{-(\vec{w}\cdot\vec{x}+b)}}$$

"logistic regression"

$e \approx 2.7$

Interpretation of logistic regression output

$$f_{\vec{w},b}(\vec{x}) = \frac{1}{1 + e^{-(\vec{w}\cdot\vec{x}+b)}}$$

"probability" that class is 1

Example:

x is "tumor size"

y is 0 (not malignant)

or 1 (malignant)

$$f_{\vec{w},b}(\vec{x}) = 0.7$$

70% chance that y is 1

P(1)

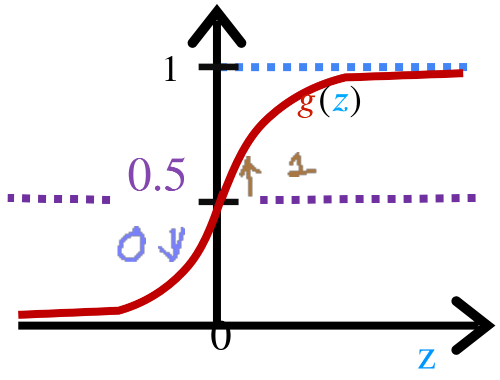
$$f_{\vec{w},b}(\vec{x}) = P(y = 1|\vec{x};\vec{w},b)$$

Probability that y is 1,
given input $\vec{x}$, parameters $\vec{w},b$

$$P(y = 0) + P(y = 1) = 1$$

Classification

Decision Boundary



$$f_{\vec{w},b}(\vec{x})$$

$$z = \vec{w} \cdot \vec{x} + b$$

$$g(z) = \frac{1}{1+e^{-z}}$$

$$f_{\vec{w},b}(\vec{x}) = g(\underbrace{\vec{w} \cdot \vec{x} + b}_{z}) = \frac{1}{1+e^{-(\vec{w} \cdot \vec{x} + b)}}$$

$$= P(y = 1 \mid x; \vec{w}, b) \quad 0.7 \quad 0.3$$

Is $f_{\vec{w},b}(\vec{x}) \geq \underbrace{0.5}_{\text{threshold}}$?

Yes: $\hat{y} = 1$ No: $\hat{y} = 0$

When is $f_{\vec{w},b}(\vec{x}) \geq 0.5$? $g(z) \geq 0.5$

$z \geq 0$	$z < 0$
$\vec{w} \cdot \vec{x} + b \geq 0$	$\vec{w} \cdot \vec{x} + b < 0$
$\hat{y} = 1$	$\hat{y} = 0$

Decision boundary

$$f_{\vec{w},b}(\vec{x}) = g(z) = g(w_1x_1 + w_2x_2 + b)$$

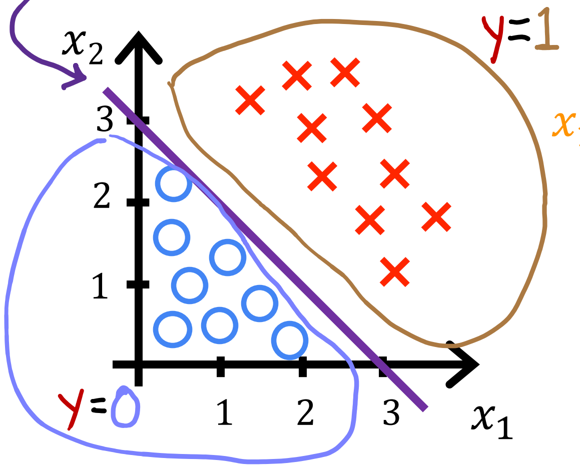
1 1 -3

Decision boundary $z = \vec{w} \cdot \vec{x} + b = 0$

$z = x_1 + x_2 - 3 = 0$

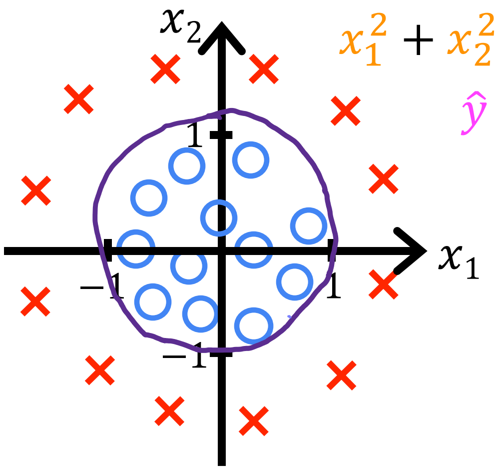
$x_1 + x_2 = 3$

$x_1 + x_2 < 3$



$x_1 + x_2 \geq 3$

Non-linear decision boundaries



$x_1^2 + x_2^2 \geq 1$

$\hat{y} = 1$

$x_1^2 + x_2^2 < 1$

$\hat{y} = 0$

decision boundary

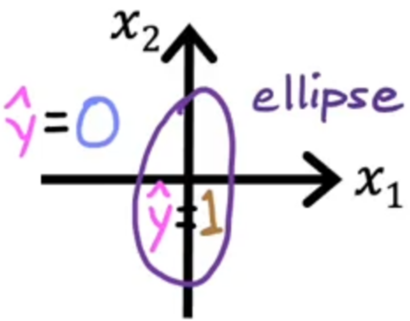
$z = x_1^2 + x_2^2 - 1 = 0$

$x_1^2 + x_2^2 = 1$

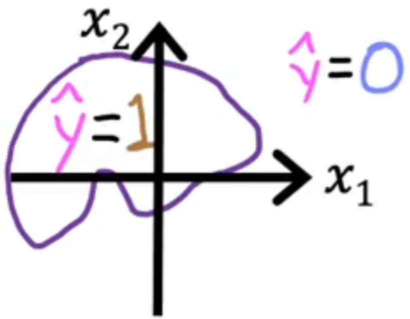
$\overbrace{w_1 x_1^2 + w_2 x_2^2 + b}^{z}$

$w_1 = 1$, $w_2 = 1$, $b = -1$

Non-linear decision boundaries



$$f_{\vec{w},b}(\vec{x}) = g(z) = g(w_1x_1 + w_2x_2$$
$$+ w_3x_1^2 + w_4x_1x_2 + w_5x_2^2$$
$$+ w_6x_1^3 + \cdots + b)$$



Cost Function

Cost Function for Logistic Regression

Training set

	tumor size (cm) x_1	...	patient's age x_n	malignant? y
$i=1$	10		52	1
⋮	2		73	0
	5		55	0
	12		49	1
$i=m$	...		...	...

$i = 1, \dots, m$ ← training examples

$j = 1, \dots, n$ ← features

target y is 0 or 1

$$f_{\vec{w},b}(\vec{x}) = \frac{1}{1 + e^{-(\vec{w}\cdot\vec{x}+b)}}$$

How to choose $\vec{w} = [w_1 \quad w_2 \quad \cdots \quad w_n]$ and b?

Squared error cost

cost

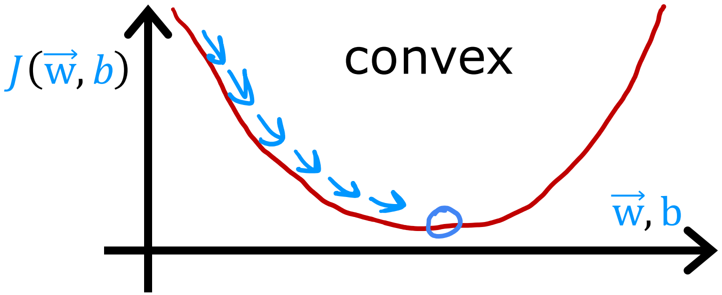
$$J(\vec{w},b) = \frac{1}{m}\sum_{i=1}^{m}\frac{1}{2}\left(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)}\right)^2$$

average of training set

loss $L\left(f_{\vec{w},b}(\vec{x}^{(i)}), y^{(i)}\right)$

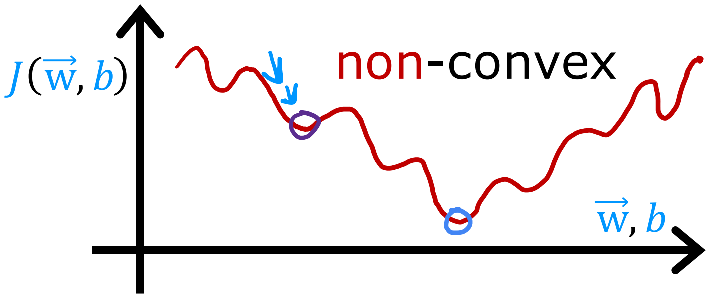
linear regression

$$f_{\vec{w},b}(\vec{x}) = \vec{w}\cdot\vec{x} + b$$



logistic regression

$$f_{\vec{w},b}(\vec{x}) = \frac{1}{1+e^{-(\vec{w}\cdot\vec{x}+b)}}$$

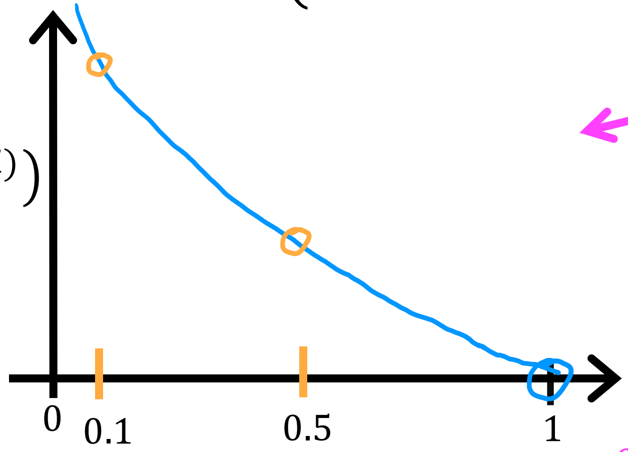


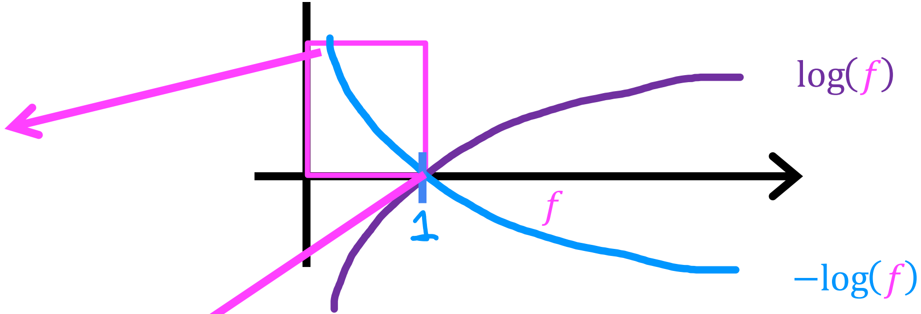
Logistic loss function

$$L\left(f_{\vec{\mathrm{w}},b}\left(\vec{\mathrm{x}}^{(i)}\right), y^{(i)}\right) = \begin{cases} -\log\left(f_{\vec{\mathrm{w}},b}\left(\vec{\mathrm{x}}^{(i)}\right)\right) & \text{if } y^{(i)} = 1 \\ -\log\left(1 - f_{\vec{\mathrm{w}},b}\left(\vec{\mathrm{x}}^{(i)}\right)\right) & \text{if } y^{(i)} = 0 \end{cases}$$

$L\left(f_{\vec{\mathrm{w}},b}\left(\vec{\mathrm{x}}^{(i)}\right), y^{(i)}\right)$

if $y^{(i)} = 1$





$f_{\vec{\mathrm{w}},b}\left(\vec{\mathrm{x}}^{(i)}\right)$

As $f_{\vec{\mathrm{w}},b}\left(\vec{\mathrm{x}}^{(i)}\right) \rightarrow 1$ then loss $\rightarrow 0$

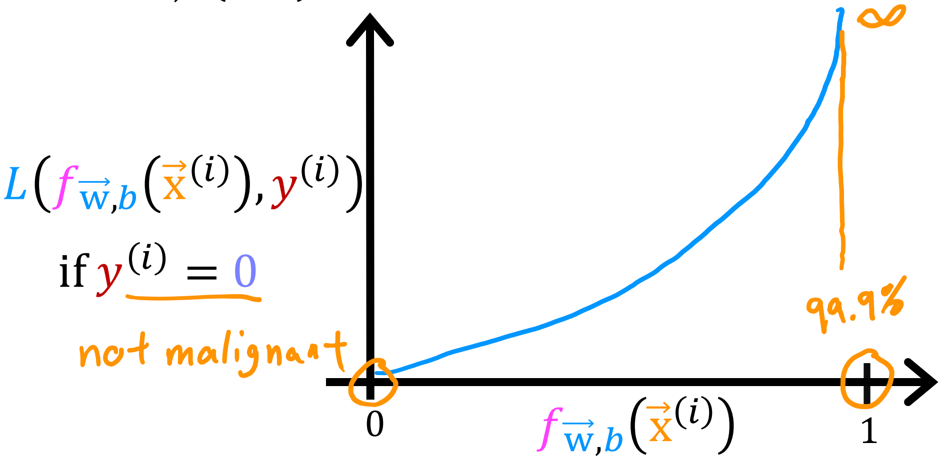
As $f_{\vec{\mathrm{w}},b}\left(\vec{\mathrm{x}}^{(i)}\right) \rightarrow 0$ then loss $\rightarrow \infty$

Loss is lowest when $f_{\vec{\mathrm{w}},b}\left(\vec{\mathrm{x}}^{(i)}\right)$ predicts close to true label $y^{(i)}$.

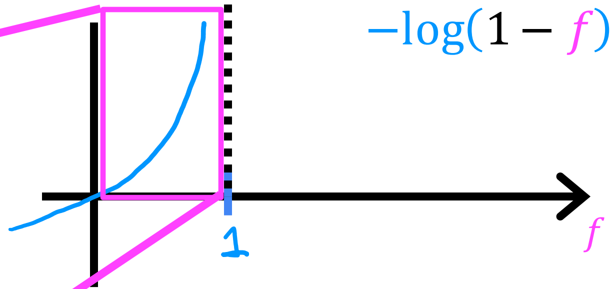
Logistic loss function

$$L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right) = \begin{cases} -\log\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) & \text{if } y^{(i)} = 1 \\ -\log\left(1 - f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) & \text{if } y^{(i)} = 0 \end{cases}$$

As $f_{\vec{w},b}\left(\vec{x}^{(i)}\right) \to 0$ then loss $\to 0$



As $f_{\vec{w},b}\left(\vec{x}^{(i)}\right) \to 1$ then loss $\to \infty$



The further prediction $f_{\vec{w},b}\left(\vec{x}^{(i)}\right)$ is from target $y^{(i)}$, the higher the loss.

Cost

$$J(\vec{w},b) = \frac{1}{m}\sum_{i=1}^{m} L\big(\underbrace{f_{\vec{w},b}(\vec{x}^{(i)}),y^{(i)}}_{\text{loss}}\big)$$

$$= \begin{cases} -\log\left(f_{\vec{w},b}(\vec{x}^{(i)})\right) & \text{if } y^{(i)} = 1 \\ -\log\left(1 - f_{\vec{w},b}(\vec{x}^{(i)})\right) & \text{if } y^{(i)} = 0 \end{cases}$$

convex
↳ can reach a global minimum

find w, b that minimize cost J

Cost Function

Simplified Cost Function for Logistic Regression

Simplified loss function

$$L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right) = \begin{cases} -\log\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) & \text{if } y^{(i)} = 1 \\ -\log\left(1 - f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) & \text{if } y^{(i)} = 0 \end{cases}$$

$$L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right) = -y^{(i)}\log\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) - \left(1 - y^{(i)}\right)\log\left(1 - f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right)$$

1 $\qquad$ $(1-1)$

if $y^{(i)} = 1$:

$$L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right) = -1 \log\left(f(\vec{x})\right)$$

Simplified loss function

$$L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right) = \begin{cases} -\log\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) & \text{if } y^{(i)} = 1 \\ -\log\left(1 - f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) & \text{if } y^{(i)} = 0 \end{cases}$$

$$L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right) = -\,y^{(i)}\log\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) - \left(1 - y^{(i)}\right)\log\left(1 - f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right)$$

0 $\quad (1-0)$

if $y^{(i)} = 1$:

$$L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right) = -1\log\left(f(\vec{x})\right)$$

if $y^{(i)} = 0$:

$$L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right) = -(1-0)\log\left(1-f(\vec{x})\right)$$

Simplified cost function

loss

$$L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right) = -y^{(i)}\log\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) - \left(1 - y^{(i)}\right)\log\left(1 - f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right)$$

cost

$$J(\vec{w}, b) = \frac{1}{m}\sum_{i=1}^{m}\left[L\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right), y^{(i)}\right)\right]$$

convex
(single global minimum)

$$= -\frac{1}{m}\sum_{i=1}^{m}\left[y^{(i)}\log\left(f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right) + \left(1 - y^{(i)}\right)\log\left(1 - f_{\vec{w},b}\left(\vec{x}^{(i)}\right)\right)\right]$$

Gradient Descent

Gradient Descent Implementation

Training logistic regression

Find $\vec{\mathrm{w}}, b$

Given new $\vec{\mathrm{x}}$, output $f_{\vec{\mathrm{w}},b}(\vec{\mathrm{x}}) = \frac{1}{1+e^{-(\vec{\mathrm{w}}\cdot\vec{\mathrm{x}}+b)}}$

$$P(y = 1|\vec{\mathrm{x}}; \vec{\mathrm{w}}, b)$$

Gradient descent

cost

$$J(\vec{w},b) = -\frac{1}{m}\sum_{i=1}^{m}\left[y^{(i)}\log\left(f_{\vec{w},b}(\vec{x}^{(i)})\right) + (1-y^{(i)})\log\left(1-f_{\vec{w},b}(\vec{x}^{(i)})\right)\right]$$

repeat {

$j=1...n$

$$w_j = w_j - \alpha\frac{\partial}{\partial w_j}J(\vec{w},b)$$

$$\frac{\partial}{\partial w_j}J(\vec{w},b) = \frac{1}{m}\sum_{i=1}^{m}(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)})x_j^{(i)}$$

$$b = b - \alpha\frac{\partial}{\partial b}J(\vec{w},b)$$

$$\frac{\partial}{\partial b}J(\vec{w},b) = \frac{1}{m}\sum_{i=1}^{m}(f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)})$$

} simultaneous updates

Gradient descent for logistic regression

repeat {

looks like linear regression!

$$w_j = w_j - \alpha \left[\frac{1}{m} \sum_{i=1}^{m} (f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)}) x_j^{(i)} \right]$$

$$b = b - \alpha \left[\frac{1}{m} \sum_{i=1}^{m} (f_{\vec{w},b}(\vec{x}^{(i)}) - y^{(i)}) \right]$$

} simultaneous updates

Same concepts:

- Monitor gradient descent (learning curve)
- Vectorized implementation
- Feature scaling

Linear regression $f_{\vec{w},b}(\vec{x}) = \vec{w} \cdot \vec{x} + b$

Logistic regression $f_{\vec{w},b}(\vec{x}) = \dfrac{1}{1 + e^{(-\vec{w} \cdot \vec{x} + b)}}$